Eur. Phys. J. B **50**, 291–294 (2006)
DOI: 10.1140/epjb/e2006-00055-7

The European Physical Journal B

q-deformed Fermions

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Received 21 October 2005 / Received in final form 18 November 2005
Published online 16 February 2006 –

Abstract. This is a study of q-Fermions resulting from q-deformed algebra of harmonic oscillators arising from two distinct algebras. Employing the first algebra, the Fock states are constructed for the generalized Fermions obeying Pauli exclusion principle. The distribution function and other thermodynamic properties such as the internal energy and entropy are derived. Another generalization of fermions from a different q-deformed algebra is investigated which deals with q-fermions not obeying the exclusion principle. Fock states are constructed for this system. The basic numbers appropriate for this system are determined as a direct consequence of the algebra. We also establish the Jackson Derivative, which is required for the q-calculus needed to describe these generalized Fermions.

PACS. 02.20.Uw Quantum groups – 03.65.-w Quantum mechanics – 05.30.-d Quantum statistical mechanics – 05.90.+m Other topics in statistical physics, thermodynamics, and nonlinear dynamical systems

1 Introduction

We shall investigate q-deformed Fermions arising as a consequence of the q-deformed algebra [1] of harmonic oscillators. We shall study two distinct algebras. First we shall consider generalized Fermions obeying the algebra $a^\dagger + q^{-1} a^\dagger a = q^{-N}$, $0 \le q \le 1$ which will be shown to obey the exclusion principle, with the Fock states restricted to $n = 0, 1$ only. This algebra is not associated with basic numbers and require the use of ordinary derivatives rather than the Jackson Derivative (JD) of q-calculus. We shall investigate in detail, the statistical thermodynamics of these Fermions. Despite the fact that they obey the exclusion principle, the thermodynamic properties are quite different from that of ordinary Fermions.

We shall also investigate q-deformed Fermions arising from the oscillator algebra $aa^\dagger + q\, a^\dagger a = q^{-N}$, $0 \le q \le 1$. It will be shown that these generalized Fermions do not obey the exclusion principle and the Fock states consist of $n = 0, 1, 2, 3, \cdots$ with arbitrary number of quanta. We shall not investigate the thermodynamics of these Fermions, but confine ourselves to a study of the Fock states and some general properties. We shall establish the JD needed for the q-calculus governing this system.

2 q-Fermions obeying Exclusion principle

Let us begin with the algebra defined by

$$aa^\dagger + q^{-1} a^\dagger a = q^{-N}, \quad 0 \le q \le 1, \tag{1}$$

which reduces to the standard Fermi algebra in the limit $q \to 1$, together with relations $[N, a] = -a, \quad [N, a^\dagger] = a^\dagger$, where $a, a^\dagger$ are the annihilation and creation operators, N is the number operator and q is the deformation parameter, which is a c-number. Let us introduce the operator $a^\dagger a = \hat{N}$, with the concomitant action on Fock states, $\hat{N}|n\rangle = \beta_n |n\rangle$, where the eigenvalue depends on n. The relation $\hat{N} a^\dagger + q^{-1} a^\dagger \hat{N} = a^\dagger q^{-N}$ follows from the algebra in equation (1). We may set $a|n\rangle = C_n |n-1\rangle$, $a^\dagger |n\rangle = C'_n |n+1\rangle$, where the constants C_n, C'_n can be determined. As a consequence we immediately obtain the recurrence relation

$$\beta_{n+1} = q^{-n} - q^{-1} \beta_n. \tag{2}$$

We may choose $\beta_0 = 0$, thus defining the ground state as vacuum. We accordingly obtain the solution

$$\beta_n = 0,\ 1,\ 0,\ q^{-2},\ 0,\ q^{-4},\ \cdots \quad = \frac{1 - (-1)^n}{2}\, q^{-n+1}. \tag{3}$$

The action of the creation and annihilation operators on the Fock states yields the results

$$a^\dagger |0\rangle = \sqrt{\beta_1} |1\rangle = |1\rangle; \quad a^\dagger a^\dagger |0\rangle = \sqrt{\beta_1}\sqrt{\beta_2} |2\rangle = 0, \tag{4}$$

the sequence of states thus terminates and consequently the Fock states are $|0\rangle$, $|1\rangle$ only. The generalized Fermions thus obey Pauli exclusion principle, just as ordinary Fermions do. As this algebra is not related to basic numbers [2], this formulation of q-fermions would employ the ordinary derivatives of thermodynamics rather than that of q-calculus. This is in contrast to the q-Fermions [3] where q-calculus analogous to q-Bosons is assumed.

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3 Thermostatics of q-fermions

From the definition of the expectation value

$$\hat{n} = \frac{1}{\mathcal{Z}}\mathrm{Tr}(e^{-\beta H}\hat{N}) = \frac{1}{\mathcal{Z}}\mathrm{Tr}(e^{-\beta H}a^{\dagger}a), \qquad (5)$$

and from the form of the Hamiltonian $H = \sum_i \hat{N}_i\,(E_i{-}\mu)$, with the operator $\hat{N}$ in the state i, we can determine the distribution function. It should be noted that $\hat{N}$ depends on q, the deformation parameter. Using the cyclic property of the trace and the relations $af(N) = f(N+1)a$, valid for any polynomial function $f(N)$, we obtain the result

$$\hat{n}_i = \frac{q^{-\hat{n}_i}}{e^{\beta(E_i-\mu)} + q^{-1}}\,. \qquad (6)$$

We shall henceforward drop the hat notation for simplicity. Using equation (2), this may be re-expressed to obtain the distribution function in the form

$$n_i = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{q^{-1}}{e^{\beta(E_i-\mu)} + q^{-1}}}\right). \qquad (7)$$

Recalling that the Fock states reduce to $n = 0, 1$ only, we observe that $\sin^2 n\pi/2 = 0, 1$ which can therefore be replaced by n without losing generality. Consequently the distribution function reduces to the simple form

$$n_i = \frac{q^{-1}}{e^{\beta(E_i-\mu)} + q^{-1}}. \qquad (8)$$

We have thus taken advantage of the exclusion principle to put the above equation in a simple form. Following standard procedure [4], we may now proceed to investigate the thermostatistics of q-Fermions as follows. The logarithm of the partition function is

$$\ln \mathcal{Z} = \sum_i \ln(1 + q^{-1}ze^{-\beta E_i}), \qquad (9)$$

which reproduces the form in equation (8), namely

$$n_i = z\frac{\partial}{\partial z}\ln \mathcal{Z} = \frac{q^{-1}}{e^{\beta(E_i-\mu)} + q^{-1}}. \qquad (10)$$

Replacing the sum over states by an integration and introducing the thermal wavelength, $\lambda = h/\sqrt{2\pi mkT}$ in the standard manner [4], we determine the expression for the thermodynamic potential

$$\Omega = -\frac{1}{\beta}\ln \mathcal{Z} = -\frac{1}{\beta\lambda^3}\ln(1 + q^{-1}z) - \frac{1}{\beta\lambda^3}f_{5/2}(q, z), \qquad (11)$$

where the generalized Riemann Zeta function f_n is defined by

$$f_n(qz) = \sum_{r=1}^{\infty}(-1)^{r+1}\frac{(qz)^r}{r^n}. \qquad (12)$$

The pressure and the mean density of the q-Fermions are determined in the thermodynamic limit as:

$$P = \lim_{V\to\infty, N\to\infty}\left(-\frac{\Omega}{V}\right) = \frac{1}{\beta\lambda^3}f_{5/2}(q^{-1}z),$$
$$\frac{N}{V} = \frac{1}{\lambda^3}f_{3/2}(q^{-1}z), \qquad (13)$$

where $v = V/N$. In the standard notation, we obtain the virial expansion

$$\frac{Pv}{kT} = 1 + \frac{1}{2^{5/2}}\left(\frac{\lambda^3}{v}\right) + \left(\frac{1}{8} - \frac{2}{3^{5/2}}\right)\left(\frac{\lambda^3}{v}\right)^2 + \cdots. \qquad (14)$$

The virial coefficients are independent of q, hence do not show deformation. Indeed it is the same as for ordinary fermions and differs from reference [3]. The internal energy and the entropy are given by

$$U = \frac{3kTV}{2\lambda^3}f_{5/2}(q^{-1}z),$$
$$\frac{S}{Nk} = \frac{5}{2}\frac{f_{5/2}(q^{-1}z)}{f_{3/2}(q^{-1}z)} - \ln z. \qquad (15)$$

It is easily verified that all of these properties have the correct Fermi limit when $q \to 1$. We shall now state some further general results for the q-fermions.

In the limit of large energy, the distribution function reduces to $n_i \longrightarrow q^{-1}e^{-\beta E_i}$, which, other than the normalization factor, describes the quantum Boltzmann statistics. In the limit when $E = \mu$, the distribution reduces to

$$n_i = \frac{q^{-1}}{1 + q^{-1}} \geq \frac{1}{2}, \qquad (16)$$

which takes the value $\frac{1}{2}$ only in the Fermi limit when $q = 1$. In the low temperature limit, when $T \to 0$, it is clear from equation (8) that the distribution function reduces to the standard unmodified step form for all values of q. Hence the effect of the deformation may be interpreted solely as a finite temperature effect.

The dependence on the parameter q is somewhat subtle for many of the thermodynamic functions and it is worthwhile discussing this. As an illustration, we shall examine the chemical potential in some detail. The number density is given by the distribution function

$$\frac{N}{V} = \frac{1}{\lambda^3}f_{3/2}(q^{-1}z), \qquad (17)$$

which can be expressed by the series

$$\frac{N}{V} = \frac{4\pi}{3}\left(\frac{2mkT}{h^2}\right)^{3/2}(\ln(q^{-1}z)^{3/2})$$
$$\times\left(1 + \frac{\pi^2}{8}(\ln(q^{-1}z)^{-2}) + \cdots\right), \qquad (18)$$

and may be employed to determine the chemical potential μ of q-fermions in terms of the Fermi-energy E_F of standard Fermions. In the lowest approximation, we obtain the result

$$\mu = E_F - kT \ln q^{-1}, \quad E_F = \frac{3N}{4\pi g V}^{2/3} \frac{h^2}{2m}. \tag{19}$$

which shows that the q-dependence appears only at finite temperatures. The expression beyond the zeroth approximation is given by

$$\mu = -kT \ln q^{-1} + E_F \left(1 - \frac{\pi^2}{12}\left(\frac{kT}{E_F}\right)^2 + \cdots\right), \tag{20}$$

which shows that the chemical potential of q-fermions is different from that of standard Fermions for $q \neq 1$.

4 Parthasarathy-Viswanathan algebra

We shall now examine the algebra $ff^\dagger + qf^\dagger f = q^{-N}$, introduced by Parthasarathy and Viswanathan [5], together with the relations $[N, f] = -f \; [N, f^\dagger] = f^\dagger$. This algebra has also been discussed by Chaichian et al. [6] and it has the expected Fermi limit when $q \to 1$. Let the operator $\tilde{N} = f^\dagger f$ act on the Fock states $|\,n\rangle$ so that $\tilde{N}|\,n\rangle = \alpha_n \tilde{N}|\,n\rangle$, where the eigenvalue depends on n. We see that the relation $\tilde{N} f^\dagger + q f^\dagger \tilde{N} = q^{-N} f^\dagger$ follows directly from the algebra. If we take $f|\,n\rangle = C_n|\,n-1\rangle, \quad f^\dagger|\,n\rangle = C'_n|\,n+1\rangle$, where C_n, C'_n are constants, we immediately obtain the relation $\alpha_{n+1} = q^{-n} - q\,\alpha_n$ for any n. Solving this recurrence relation, we accordingly determine α_n to be

$$\alpha_0 = 0,\; \alpha_1 = 1,\; \alpha_2 = q^{-1} - q,\; \cdots, \\ \alpha_n = q^{-n+1} - q^{-n+3} + \cdots q^{n-3} - q^{n-1}, \tag{21}$$

which is immediately recognized as the basic number,

$$\alpha_n = [n] = \frac{q^{-n} - (-1)^n q^n}{q + q^{-1}}, \tag{22}$$

appropriate for q-fermions [5] and the solution of the recurrence relation above indeed explains how $f^\dagger f = [N]$ is a direct consequence of the algebra. We further obtain the result $ff^\dagger = [N+1]$. We observe that Pauli exclusion principle is valid only in the limit $q \to 1$. For arbitrary values of q, we may construct the Fock states according to

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{[n]!}}\,|0\rangle, \tag{23}$$

where $[n]! = [n] \cdot [n-1] \cdots [1]$. The exclusion principle follows when $q \to 1$ since $\alpha_2 = 0$. Hence we need not assume the relation $f^2 = (f^\dagger)^2 = 0$ as in ref.[6] for invoking the exclusion principle in the limit. These are generalized fermions with $|n\rangle,\; n = 0, 1, 2, \cdots$ when $q \neq 1$. However, $[n] = \frac{1}{2}(1 - (-1)^n)$ in the limit $q \to 1$ and the Fock space breaks up into an infinity of 2-dimensional subspaces when $q = 1$, with the Pauli principle valid in each subspace. The basic number here exhibits skew symmetry i.e., $[n] \longrightarrow \pm[n]$ for n =odd, even and this contrasts with the situation in other algebras. Thus special care is needed in order to identify q-calculus with JD in this formalism which determines the thermodynamics of these generalized Fermions. For this purpose, we proceed to analyze as follows.

First, we recall the JD in the q-boson case [3] which reduces to the ordinary derivative in the limit $q \to 1$. To study the case of q-fermions, arising from the algebra $ff^\dagger + qf^\dagger f = q^{-N}$, we may invoke the holomorphy relation $f \iff \mathcal{D}_x, \;\; f^\dagger \iff x$. The algebra thus implies $\mathcal{D}_x x + qx\mathcal{D} = q^{-N}$. It may be useful to recall that the holomorphy leads to properties [7] such as

$$q^N x = x q^{N+1};\; q^N x^r = (qx)^r \quad [N]x = x[N+1], \\ [N]x + qxN = xq^{-N}, \tag{24}$$

etc. Consequently, we infer the solution of $\mathcal{D}_x x + qx\mathcal{D} = q^{-N}$ to be

$$\mathcal{D}_x = \frac{1}{x}\,\frac{q^{-N} - (-1)^N q^N}{q + q^{-1}} \tag{25}$$

as the appropriate JD for q-fermions. If we now employ the properties

$$q^N f(x) = f(qx), \quad q^{-N} f(x) = f(q^{-1}x), \\ (-q)^N f(x) = f(-qx), \tag{26}$$

this can be expressed as a differential operator in the standard manner,

$$\mathcal{D}_x\; f(x) = \frac{1}{x}\,\frac{f(q^{-1}x) - f(-qx)}{q + q^{-1}}, \tag{27}$$

valid for q-fermions. One can investigate many of the properties [8] satisfied by the JD. In particular the q-Fermion JD does not reduce to the ordinary derivative when $q \to 1$.

5 Summary and conclusion

We have investigated the consequences of the q-deformed algebra $a^\dagger + q^{-1}a^\dagger a = q^{-N},\; 0 \leq q \leq 1$ describing generalized Fermions obeying the exclusion principle which reduces to the ordinary Fermions in the limit $q \to 1$. In addition to the mathematical formulation, we have presented detailed physical applications of the generalized Fermions and determined the various thermodynamic functions such as the partition function, pressure, and the entropy. We have also determined the dependence on q of the chemical potential as a function of temperature. This is an example where the deformation is seen to be a finite temperature effect. The Fock states are constructed by $|n\rangle = (a^\dagger)^n/\sqrt{\beta_n}\,!|0\rangle$, where β_n depends on q and $\beta_n = 0, 1$ for $n = 0, 1$. While the thermodynamic properties of these Fermions are dependent on the deformation parameter, the algebra nevertheless has no basic numbers associated with it and the system is governed by the ordinary calculus of thermodynamics, in contrast to the earlier work cited [3].

We have also investigated the generalized Fermions, not obeying the exclusion principle, stemming from another q-deformed oscillator algebra. We have established the following basic premises. The Fock states of these generalized Fermions can be built from the action of the creation operators and require the use of Fermion basic numbers which follow directly from the algebra. The q-calculus needed to study the thermostatics of these Fermions must employ a JD which is characteristic of the nature of the generalized Fermions. We have determined the form of this JD.

The author would like to thank A. Lavagno and A. Scarfone of Politecnico di Torino, Torino, Italy, for valuable discussions on the subject of q-deformed algebras.

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